BULLETIN

OF THE

Society for the

Promotion of Engineering Education

Vol. I LANCASTER, PA., APRIL, 1911 No. 9

EDITORIAL.

The Mathematics Report.—A great deal of space in the BULLETIN is devoted to the report of the Committee on the Teaching of Mathematics to Engineering Students. Every effort has been made to bring this report prominently to the attention of the membership of the Society. The preliminary report was distributed in galley form at the Madison meeting. Since that meeting it has been revised by the chairman after conferences and discussion with members of the committee and others. In the form in which it appears in the BULLETIN it is still a preliminary report, but it is evidently taking on more and more finished form. The chairman of the committee has labored long and hard on this report and he deserves the cordial support of the membership. If this report can be acted upon definitely at the Pittsburgh meeting it will then begin a most important work in engineering education.

The Report on Engineering Degrees.—This important report will appear in Volume 18 of the PROCEEDINGS as amended slightly by the Madison meeting. It is suggested that copies of the preliminary report, which appeared in the supplement to the June BULLETIN, be destroyed. The changes, although slight, are very important. The report is being reprinted by the Society in pamphlet form for use in those quarters where it will do the most good in encouraging the adoption of logical degrees.

The Proceedings.—Volume 18 of the PROCEEDINGS is now off the press and in process of distribution. In response to the query, "Why is it so late?" the attention of the members is called to the present status of this publication. The publication of the BULLETIN this year gave, for the first time, the opportunity to place before the entire membership many of the most important papers. The result has been that a large number of very carefully-written contributions to the discussion have been received. These are indicated in the volume by the expression "by letter." The publication of papers was continued into December so that the largest possible number could be printed. After allowing a few weeks for discussion the matter was edited and sent to the printer about the end of December. Since that time the work has been going on steadily. Members are requested to return promptly the receipt postal cards enclosed with the volumes. These serve to correct the addresses as well as to indicate that the volumes arrived in good condition. This small matter is apt to be overlooked. A receipt card for 1906 arrived at the office a few weeks ago.

New Members.—The Membership Committee appointed some weeks ago is addressing invitations to a number of prominent teachers and engineers. A number of these are enrolling. The Executive Committee felt that it was not just to charge the full fee for the period remaining before the annual meeting and they have therefore made a special fee of $2.50. This will entitle each new member to the BULLETIN from January to June, to the PROCEEDINGS (nearly 500 pages), and to such other privileges as are enjoyed by the membership.

APPLICANTS FOR MEMBERSHIP.

CONVERSE, C. W., Asst. Professor of Electrical Engineering, University of Oregon, Eugene, Ore. 1911

HESS, HENRY, President, the Hess-Bright Mfg. Co., and the Hess Steel Castings Co., Philadelphia, Pa. 1911

HUNT, C. W., Secretary, the American Society of Civil Engineers, 220 W. 57th St., New York, N. Y. 1911
McCOARD, G. W., Professor of Mathematics, The Ohio State University, Columbus, Ohio..... 1911
SMITH, A. B., Electrical Engineer, The Automatic Electric Co. Evanston, Ill. 1911

(A large number of applications received too late for insertion here will be acted upon and the names will be printed in the May BULLETIN.)

CHANGES IN ADDRESSES.

Biegler, P. S. (6 Riverside Ave., Spokane, Wash.); Gibson, A. J. (Professor of Engineering, University of Queensland, Brisbane, Queensland, Australia); Grover, N. C. (1330 F. St., N. W., Washington, D. C.); LeBaron, J. F. (Consulting Engineer in General Practise, Charden, Ohio); Mendenhall, T. C. (mail should be addressed care of T. E. Mendenhall, Madison, Wis.); Molitor, D. A. (has resigned professorship of civil engineering at Cornell University and is now located at 1012 Baltimore Ave., Kansas City, Mo.); Munroe, J. P. (mail should be addressed to 77 Summer St., not M. I. T., Boston, Mass.).

PITTSBURGH CONVENTION COMMITTEE.

President Talbot has appointed the following members as a local committee:

J. H. Leete, Dean of the School of Applied Science, Carnegie Technical Schools, Chairman.

F. L. Bishop, Dean of the School of Engineering, University of Pittsburgh.

Clifford Connelley, Dean of the School of Applied Industry, Carnegie Technical Schools.

Wm. E. Mott, Professor of Civil Engineering, Carnegie Technical Schools.

Chas. F. Scott, Consulting Engineer, Westinghouse Electric and Manufacturing Co.

A report from this committee will probably be published in the May BULLETIN.

LETTER TO THE EDITOR.

DEUTSCHER AUSSCHUSS FÜR TECHNISCHES SCHULWESEN.

CHARLOTTEN STR. 43, BERLIN, NW. 7,
Jan. 25, 1911.

SECRETARY S. P. E. E.,
ITHACA, N. Y.,

Dear Sir: I thank you very much for your esteemed communication of January 13, and for the accompanying copies of your publication which came to hand. The German Committee will certainly note these articles with great interest, and as far as possible make use of the suggestions expressed therein.

We shall not fail to send you our publications regularly. It is expected that volume II. of writings of the Deutscher Ausschusses will appear in April, and will then be forwarded to you.

The Deutscher Ausschuss is particularly occupied at present with questions of technical instruction in the high schools, as well as with problems in shop instruction. Furthermore, in the near future, we shall have to look very carefully to the practical training of our engineers. In general, we require that the students of a technical high school have worked at least one year in a manufacturing establishment. We require of the scholars of the technical preparatory schools, at least a two-year period of practical work. It is directly upon this practical experience that very great value is attached by the industries. It is desired to obtain by means of this period of practical work not only an intimate knowledge of the materials and their manufacture, but also since the students are at work in the midst of a large establishment, they should thereby learn to understand some of the social and commercial relations as well. Since the latter is also very desirable,

a course in the manufacturing establishments which are connected with the schools, is not considered futile.

But a great difficulty can be foreseen therein, for even to-day it is already hard to find suitable firms to accept young men for the purpose of instructing them. Experience also proves that these young men are often left too much to themselves, so that it is then complained that too little result is achieved during the period of practical work.

In every case we intend to study all these relations carefully and we would learn with interest of the relations as found in the United States. We would therefore be very obliged to you, if you will be so kind as to inform us about these matters, or send us the addresses through which we could perhaps get sufficient information.

At your service to return the favor, I am,

Yours most respectfully,

(Signed) C. MATSCHOSS.

Members are requested to send Mr. Matschoss any suggestions regarding this matter. This is an excellent way to promote international friendliness.—ED.

EDITORIAL NOTES.

Professor Talbot writes that an informal committee, with representatives of five departments including mathematics, has been made up at the University of Illinois with a view to having a discussion of the report of the Mathematics Committee.

A. C. McClurg and Co. have reprinted their booklet comprising the last approved report of the Committee on Engineering Books. To this have been added the lists prepared by the Pratt Institute Library. The booklet plainly states that this is not the revised report of the Committee. This last report is now in the hands of the Publication Committee.

COLLEGE NEWS.

Brown University.—On Visitation Day, February 22, President W. H. P. Faunce, of Brown University, announced that the Elmer L. Corthell Engineering Library had been given to the University by Dr. Elmer L. Corthell, Brown '67, M. Am. Soc. C. E. This library subscribes to sixty-four current engineering periodicals, including proceedings of engineering societies and engineering magazines. The library will be housed in a special room in the new John Hay Memorial Library which was dedicated on November 10, 1910.—On March 9 Professor G. W. Ritchie, of the Mount Wilson Observatory, Cal., delivered an address before the Brown Chapter of the Society of Sigma Xi on "Recent Celestial Photography with the Sixty-inch Reflector at the Mount Wilson Observatory."

Carnegie Technical Schools.—The architects are now at work upon the plans for the additional buildings made possible by Mr. Carnegie's recent gift. The new buildings include a Head House for the School of Applied Science, an addition to Machinery Hall, a new Hydraulic Laboratory, a building for the School of Applied Design, an addition to the Women's School, an Administration Building, a Gymnasium and a Student Commons.—Professor W. E. Mott, head of the Department of Civil Engineering, is with the American Society of Civil Engineers on their trip to Panama.—The Department of Metallurgy is now installing equipment for electric furnace work, a field which is particularly important in this district.

Clarkson School of Technology.—At the Charter Day Exercises of the fifteenth anniversary, on March 17, Mr. M. N. Baker, associate editor of *Engineering News,* delivered an address on "The Engineer and Social Service." The School is to have a new gymnasium by September, 1911.

State Agricultural College of Colorado.—Professor E. B. House succeeds Professor L. G. Carpenter as head of the Civil Engineering Department, the latter having resigned to enter practical work.

University of Illinois.—Mr. D. A. Abrams, of the Materials Testing Laboratory of the University of Illinois, exhibited at the Convention of Concrete Tile Manufacturers, held in Chicago at the time of the Cement Show, a testing machine for testing the strength of drain tile. This machine was designed by him and built in the shops of the Materials Testing Laboratory. A paper on the testing of drain tile was presented by Professor A. N. Talbot and Mr. Abrams.—The Illinois Legislature made an official visit of inspection to the State University on February 16. The College of Engineering is asking for buildings to relieve the overcrowded conditions of its laboratories and class rooms, especially the laboratory space devoted to railway engineering and materials testing.

The Ohio State University.—On Tuesday evening, February 21, Col. Geo. W. Goethals, chief engineer of the Panama Canal, spoke before the Omega Chapter of the Society of Sigma Xi on the Panama Canal, and illustrated the lecture with about 250 lantern slides and several sets of moving pictures illustrating the work in operation.—On February 22, exercises were held in the University Chapel commemorative of the life and services of Stillman Williams Robinson, C.E., D.Sc., professor of mechanical engineering and physics, 1878–1881, professor of mechanical engineering, 1881–1895, emeritus professor of mechanical engineering, 1899–1910. The speakers were: Professor Mortimer E. Cooley, dean of the Engineering Department, who represented the University of Michigan and read a paper prepared by Professor Joseph Baker Davis, giving an account of Professor Robinson's life as Professor Davis knew him at the University of Michigan; Professor Ira Osborn Baker represented the University of Illinois, where Professor Robinson founded the Department of Mechanical Engineering. Those who had visited the present extensive laboratories of the University of Illinois were

able to appreciate the retrospective of Professor Robinson's first equipment of a few tools housed in a mule shed. Mr. Charles Frederick Marvin, of the class of 1883, and professor of meteorology of the U. S. Weather Bureau, spoke in behalf of the alumni of the department. Professor Embury Asbury Hitchcock, who had been assistant to Professor Robinson in the department and a resident at his house, spoke in behalf of the faculties of Ohio State University. The four addresses very completely covered his life and described his genius as a mechanic, mathematician, scientist and an engineer, and withal a man who by patience and good works manifested his love to colleague and students. The four addresses were of inspirational value to those who were fortunate enough to hear them. It is expected that they will be published by the university as a bulletin.

Rensselaer Polytechnic Institute.—The current meter testing flume presented by the firm of W. and L. E. Gurley, has been enclosed and may now be used in all weather. The flume is about one hundred feet long and of a cross section of four feet by four feet. The distance passed over by car to which the meter is attached, the revolutions of the meter and the strokes of a half-second clock are recorded on a chronograph. In this way an accurate reading of velocity of car and revolutions per minute of meter may be obtained. The flume has been used by Professor S. F. Moody, of the Department of Mechanical Engineering, to determine the effect of the inclination of the velocity nozzle of a Pitot tube, the effect of vibration of water and the constant of the tube in the standard condition.—The senior theses in mechanical engineering subject include: The investigation of the relative merits of gas, gasoline and alcohol as a fuel for a given engine; the test of a producer and engine; the effect of superheat on the heat used by a small steam turbine; the design of several hydraulic plants; the design of a heating and lighting plant for an institution; the design of a cold storage plant; the design of a city water works pumping station.

A SYLLABUS OF ELEMENTARY GEOMETRY AND MENSURATION.*

This syllabus is intended to include those facts and methods of elementary geometry which a student should have so firmly fixed in his memory that he will never think of looking them up in a book.

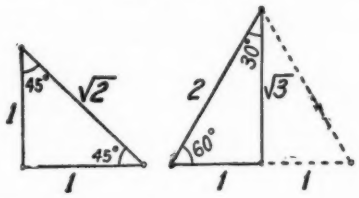
1. Right Triangles.

In a right triangle, the square on the hypothenuse is equal to the sum of the squares on the other two sides (Pythagoras, 580–501 B.C.); and the sum of the acute angles is 90°.

Examples of right triangles with integral sides: 3, 4, 5; 5, 12, 13.

Two right triangles are congruent when they agree with respect to (*a*) any side and an acute angle; or (*b*) any two sides.

In the "45° triangle" and the "30–60 triangle," the ratios of the sides are as indicated in the figure.



* This syllabus is a preliminary draft of a chapter of the Report of the Committee on the Teaching of Mathematics to Students of Engineering, prepared by the Chairman, and printed in advance in order to secure the widest criticism from all the members of the Society. (See page 407.) *The Committee as a whole is not responsible for the report in its present form.* Since elementary geometry is the subject in which the mathematical preparation of students of engineering appears to be the most nearly satisfactory, this subject in treated only briefly in this report. Criticisms and suggestions from any source are invited, and may be addressed to the Chairman, Professor E. V. Huntington, 27 Everett Street, Cambridge, Mass.

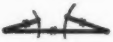
2. Oblique Triangles.

In any plane triangle, the sum of the angles is 180°. Hence, an exterior angle of a triangle equals the sum of the opposite interior angles.

Of two unequal sides in a triangle, the greater is opposite the greater angle.

A plane triangle is, in general, wholly determined when any three of its parts (not all angles) are given.

There are four cases:



(*a*) two angles (provided their sum is less than 180°) and one side;

(*b*) two sides and the included angle;

(*c*) the three sides (provided the largest is less than the sum of the other two);

(*d*) two sides and the angle opposite one of them (the "ambiguous case," in which we may have two solutions, or one, or none).

Hence the usual rules for testing the equality of two plane triangles.

The center of gravity of a plane triangle is the intersection of the three medians, and is two thirds of the way from any vertex to the middle point of the opposite side.

3. Angles in a Circle.

An angle inscribed in a semicircle is a right angle.

An angle subtended by an arc of a circle at any point of the circumference is equal to half the angle subtended by the same arc at the center.

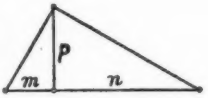
4. Similar Figures. Proportion.

If any two lines are cut by a set of parallels, the corresponding segments are proportional. (Hence the usual rule for dividing a given line into any number of equal parts.)

In all problems in proportion, the notation $a:b::c:d$, and all special terminology, such as "taking a proportion by alternation," "by composition," etc., should be abandoned in favor of the ordinary language of the equation. For example, if $a/b = c/d$, then, by adding 1 to both sides, $(a+b)/b = (c+d)/d$; and by subtracting 1 from both sides, $(a-b)/b = (c-d)/d$; etc.

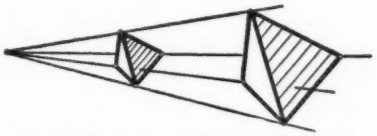
If two plane triangles are similar, their corresponding sides are proportional.

In a right triangle, the perpendicular from the vertex of the right angle to the hypothenuse is a mean proportional between the segments of the hypothenuse:



$$p^2 = mn.$$

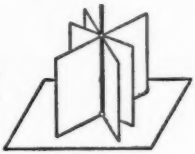
Any two similar figures, in the plane or in space, can be placed in "perspective," that is, so that lines joining corresponding points of the two figures will pass through a common point. In other words, of two similar figures, one is merely an enlargement of the other.



In two similar figures, if k is the factor of proportionality, any *length* in one $= k \times$ (the corresponding *length* in the other); any *area* in one $= k^2 \times$ (the corresponding *area* in the other); any *volume* in one $= k^3 \times$ (the corresponding *volume* in the other).

5. Lines and Planes.

If a line is perpendicular to a plane, every plane containing that line is perpendicular to the plane.



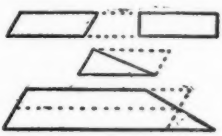
A dihedral angle is measured by a plane angle formed by two lines, one in each face, perpendicular to the edge.

6. Plane Areas.

Area of parallelogram
$=$ base $\times$ altitude.

Area of triangle
$= \frac{1}{2}$ base $\times$ altitude.

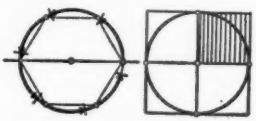
Area of trapezoid
$= \frac{1}{2}$ sum of $\parallel$ sides $\times$ alt.
$=$ mid-section $\times$ altitude.



7. The Circle. ($\pi = 3.1416 \cdots = 22/7$, approximately.)

Circumference of circle $= 2\pi r$.

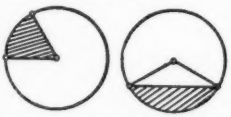
(Proved by regarding the circle as the limit of an inscribed or circumscribed polygon; proof rather long.)



Area of circle $= \pi r^2$.

(Proof by regarding circle as limit of sum of triangles radiating out from the center, the altitude of each triangle being the radius of the circle; hence, area of circle $= \frac{1}{2}$ circumference $\times$ radius.)

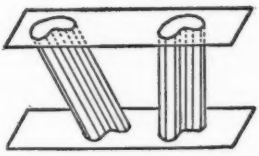
$$\frac{\text{Area of sector}}{\text{area of circle}} = \frac{\text{angle of sector}}{\text{four right angles}}\text{; hence,}$$



Area of sector $= \frac{1}{2}r^2\theta$, where θ is the angle in radians.

For area of segment, subtract triangle from sector.

8. The Cylinder.

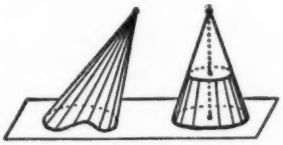


Volume of any cylinder (or prism) = base × altitude.

Area of curved surface of any *right* cylinder (or *right* prism) = perimeter of base × altitude.

(Proof by regarding the area as the limit of a sum of rectangles whose common altitude is the altitude of the cylinder; or, by slitting the cylinder along an "element" and rolling the surface out into a rectangle.)

9. The Cone.



Volume of any cone (or pyramid) = 1/3 base × altitude.

(Proof by dissecting a triangle prism; or, more simply, by the integral calculus.)

Area of curved surface of a *right* circular cone (or a *regular* pyramid) = 1/2 perimeter of base × slant height.

(Proof by regarding the area as the limit of a sum of triangle whose common altitude is the slant height of the cone.)

Area of frustum of a right circular cone (or of a regular pyramid)

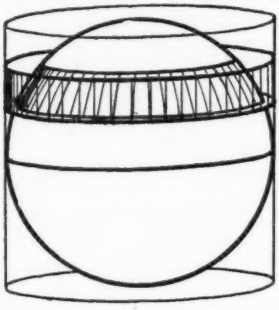
= 1/2 sum of perimeter of bases × slant height.
= perimeter of mid-section × slant height.

(Proof by regarding the area as the limit of the trapezoids whose common altitude is the slant height of the frustum.)

10. The Sphere.

Area of a zone = circumference of great circle × altitude of zone.

In other words, the area of the sphere cut out by two parallel planes is equal to the area of the portion of the circumscribing cylinder intercepting between the same pair of parallel planes. (Proof by regarding the zone as the limit of a sum of conical frustums.) Hence,



Area of sphere $= 4\pi r^2$

= area of four great circles of the sphere.

In other words, the *area of the sphere is equal to the area of the curved surface of the circumscribing cylinder.*

Volume of sphere $= \frac{4}{3}\pi r^3$.

(Proof by regarding sphere as limit of a sum of pyramids radiating out from the center, the altitude of each pyramid being the radius of the sphere; hence, volume of sphere = ⅓ area of sphere × radius.)

$$\frac{\text{Area of a lune}}{\text{area of sphere}} = \frac{\text{four right angles}}{\text{angle of lune}}.$$

Area of spherical triangle is proportional to its spherical excess (that is, the excess of the sum of its angles over 180°).

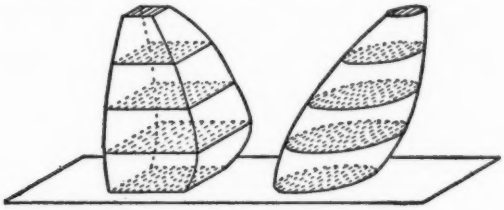
(Proof by considering three lunes which have the given triangle in common.)

11. Cavalieri's Theorem (1598–1647).

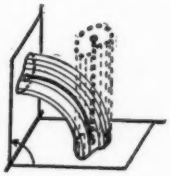
Suppose two solids have their bases in the same plane, and let the sections made in each solid by any plane parallel to the base be called "corresponding sections." If then the corresponding sections of the two solids are always equal, the volumes of the solids will be equal.

(Proof by regarding each of the solids as the limit of a pile of thin slabs.)

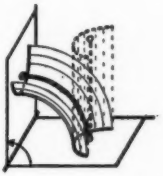
The following further theorem, the proof of which involves the integral calculus, are mentioned here also, because they are easy to remember and are often serviceable in elementary work.



12. Theorems of Guldin (1577–1643), **or of Pappus** (about 290 A.D.).



1. Suppose a plane figure revolves about an axis in its plane but not cutting it. Then the volume of the solid thus generated is equal to the area of the given figure times the length of the path traced by its center of gravity.



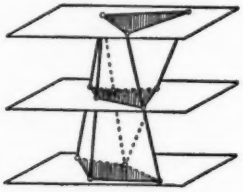
2. Suppose a plane curve revolves about an axis in its plane but not cutting it. Then the area of the surface thus generated is equal to the length of the given curve times the length of the path traced by its center of gravity.

13. The Prismoidal Formula.

The prismoidal formula holds for any solid lying between two parallel planes and such that the area of a section at distance x from the base is expressible as a polynomial of the second (or third) degree in x.

If A, B = areas of the bases, M = area of a plane section midway between the bases, and h = altitude, then

$$\text{Volume of prismoid } \frac{h}{6} = (A + B + 4M).$$



A SYLLABUS OF ANALYTIC GEOMETRY.*

This syllabus is intended to include those facts and methods of analytic geometry which a student who has completed an elementary course in that subject should have so firmly fixed in his memory that he will never think of looking them up in a book.

A course of study in analytic geometry should consist chiefly of *problems* solved by the students, each problem being solved in the basis of a small number of fundamental formulas such as are here mentioned.

This syllabus is confined mainly to the conic sections; but a satisfactory course in analytic geometry should include also the study of many other curves, both in rectangular and in polar coordinates. The syllabus takes up only those properties of curves which can be readily investigated without the aid of the calculus; but the present tendency to introduce the elements of the calculus *before* any elaborate study of geometry is attempted is to be much encouraged.

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* Prepared by the Chairman of the Committee on the Teaching of Mathematics to students of Engineering, for discussion at the Pittsburgh meeting of the Society. (See page 407.) *The Committee as a whole is not responsible for the Syllabus in its present form.* Criticism and suggestions from any source are invited, and may be addressed to the Chairman, Professor E. V. Huntington, 27 Everett Street, Cambridge, Mass.

A SYLLABUS OF ANALYTIC GEOMETRY.

TABLE OF CONTENTS.

Chapter I. Rectangular Coördinates.
Chapter II. The Straight Line: Equations of the form $Ax + By + C = 0$.
Chapter III. The Circle: Equations of the form $x^2 + y^2 + Dx + Ey + F = 0$.
Chapter IV. The Parabola: $y^2 = 2px$.
Chapter V. The Ellipse: $b^2x^2 + a^2y^2 = a^2b^2$.
Chapter VI. The Hyperbola: $b^2x^2 - a^2y^2 = a^2b^2$.
Chapter VII. Transformation of Coördinates.
Chapter VIII. General Equation of the Second Degree in x and y.
Chapter IX. Systems of Conics.
Chapter X. Polar Coördinates.
Chapter XI. Coördinates in Space.

CHAPTER I.

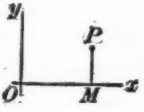
RECTANGULAR COORDINATES.

1. In many geometrical problems it is convenient to describe the position of a point in a plane by giving its distances from two fixed (perpendicular) lines in the plane.*

For example, on a map, the distance of a point to the east or west from a fixed meridian is called the longitude of the point, and its distance north or south from the equator is called its latitude.

So in general, in any plane, the distance of a point to the right or left from a fixed vertical axis is called the *abscissa*, x, of the point, and its distance up or down from a fixed horizontal axis is called its *ordinate*, y. The x and y together are called the *coordinates* of the point.

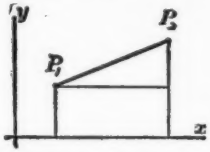
The value of $x\,(=OM)$ will be positive to the right, negative to the left; the value of $y\,(=MP)$ will be positive upward, negative downward. The point for which $x = x_1$ and $y = y_1$ is denoted by P_1, or (x_1, y_1).



* We restrict ourselves here to *rectangular* axes; oblique axes are, however, occasionally useful.

2. To express *the distance between two points* in terms of their coordinates:

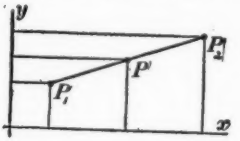
$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



3. To find the coordinates of *the point half way between two given points:*

$$x = \tfrac{1}{2}(x_1 + x_2),$$

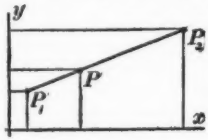
$$y = \tfrac{1}{2}(y_1 + y_2).$$



4. *To find the coordinates of a point P on the line through two fixed points,* and such that its distance from the first point is n times the distance between the two points ($\overline{P_1P} = n\overline{P_1P_2}$):

$$x = x_1 + n(x_2 - x_1),$$

$$y = y_1 + n(y_2 - y_1).$$

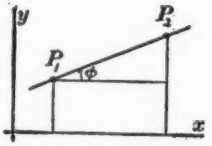


Here n may be any real number (positive, negative, or zero).

5. *To find the slope of a line through two given points:*

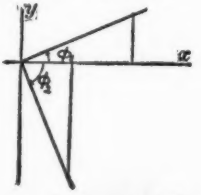
$$m = \tan\phi = \frac{y_2 - y_1}{x_2 - x_1}.$$

The angle ϕ is called the *inclination* of the line; $\tan\phi$ is the *slope*.



6. If two lines are *parallel,* their slopes are equal: $m_1 = m_2$.

If two lines are *perpendicular,* the product of their slopes is minus one: $m_1 m_2 = -1$.



7. To express *the areas of triangles and polygons* in terms of the coordinates of the vertices, consider the trapezoids formed by the ordinates drawn to the vertices.

8. In any problem involving an unknown point, remember that *two conditions* are necessary to determine the coordinates of the point (simultaneous equations in two unknown quantities).

CHAPTER II.

THE STRAIGHT LINE: EQUATIONS OF THE FORM $Ax + By + C = 0$.

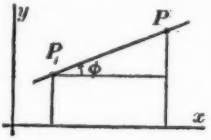
9. We have seen that if two conditions are imposed on x and y, the position of the point (x, y) is wholly determined. If only one condition is imposed, the position of the point is only partially restricted. (Examples: $x = 5$, $x^2 + y^2 = 25$, etc.)

The collection of all points which satisfy a given condition imposed on x and y is called the *locus* of that condition; and the condition itself, expressed in algebraic language, is called the *equation of the locus*. Thus, the *equation of a straight line* is the algebraic expression of the *condition* which x and y must satisfy in order that the point (x, y) shall lie on the line; in other words, *the equation of a line is an equation which is* TRUE *when the coordinates of any point on the line are substituted for x and y, and* FALSE *when the coordinates of any point off the line are substituted for x and y;* and so in general for the equation of any locus.

10. To find the coordinates of the points of intersection of two loci whose equations are given, we have simply to find the pairs of values of x and y (if any) which satisfy both the equations at once (simultaneous equations in x and y).

11. *To find the equation of a line* (not perpendicular to either axis), *when its slope, m, and the coordinates of one of its points (x_1, y_1), are given:*

$$y - y_1 = m(x - x_1).$$



The equation of a line perpendicular to the x-axis (or the y-axis) is, by inspection, $x = a$ (or $y = b$).

The equation of any straight line is of the form $Ax + By + C = 0$, and the locus of every equation of the form $Ax + By + C = 0$ is a straight line. Hence, to plot the locus of such an equation, it is sufficient to find any two of its points.

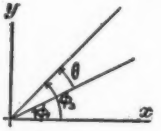
12. *To find the slope of a line whose equation is given* (the line being not perpendicular to an axis), write the equation in the form $y = (\ )x + (\ )$; then the coefficient of x will be the slope.

13. To find the equation of a line *parallel* or *perpendicular* to a given line and through a given point, remember that $m_1 = m_2$ for parallel lines, and $m_1 m_2 = -1$ for perpendicular lines (see § 6).

Special method: if the given line is $Ax + By + C = 0$, then the parallel is $Ax + By = k$ and the perpendicular is $Bx - Ay = K$, where k and K are to be determined.

14. *To find the angle θ between two lines whose slopes are given:*

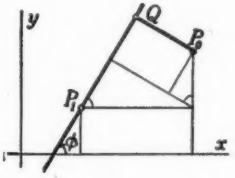
$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}.*$$



15. *To find the distance between a given point (x_0, y_0), and a given line:*

(1) When the inclination of the line, ϕ, and the coordinates of one of its points, (x_1, y_1), are given, we have from the figure:

$$\overline{QP_0} = (x_0 - x_1)\sin\phi - (y_0 - y_1)\cos\phi,$$



(2) When the equation of the line is given in the form $Ax + By + C = 0$, use the following formula:†

$$D = \left| \frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}} \right|.$$

Here the vertical bars mean "the absolute value of."

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* Proof: By trigonometry, $\tan(\phi_2 - \phi_1) = \dfrac{\tan\phi_2 - \tan\phi_1}{1 + \tan\phi_2 \tan\phi_1}$.

† Proof: Show that the foot of the perpendicular from P_0 to the line $Ax + By + C = 0$ has the coordinates $x_2 = x_0 - hA$, $y_2 = y_0 - hB$, where $h = (Ax_0 + By_0 + C) / (A^2 + B^2)$.

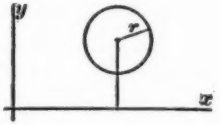
CHAPTER III.

THE CIRCLE: EQUATIONS OF THE FORM $x^2 + y^2 + Dx + Ey + F = 0$.

16. The *equation of a circle* is the algebraic expression of the *condition* which x and y must satisfy in order that the point (x,y) shall lie on the circle (see § 9 and § 10).

17. *To find the equation of a circle when its radius, r, and the coordinates (α, β) of its centre are given:*

$$(x - \alpha)^2 + (y - \beta)^2 = r^2.$$



When the centre is at the origin (0, 0), this equation becomes

$$x^2 + y^2 = r^2.$$

18. The equation of any circle is of the form $x^2+y^2+Dx+Ey+F=0$. Conversely, every equation of the form $x^2 + y^2 + Dx + Ey + F = 0$ can be reduced to the form $(x + \frac{D}{2})^2 + (y + \frac{E}{2})^2 = \frac{1}{4}(D^2 + E^2 - 4F)$, and therefore represents a circle with centre at $(-D/2, -E/2)$, or a single point, or no locus, according as $D^2 + E^2 - 4F$ is positive, zero, or negative. When we say, in brief, that the locus of any equation of the form $x^2 + y^2 + Dx + Ey + F = 0$ is a "circle," we must understand that the "circle" may be "real," "null," or "imaginary."

19. *To find the centre and radius of a circle whose equation is given,* do not use a formula, but "complete the squares" of the terms in x and y in each case, and compare with the standard equation in the manner just indicated.

20. *In problems concerning tangents to a circle,* use the fact that the tangent is perpendicular to the radius drawn to the point of contact.

CHAPTER IV.

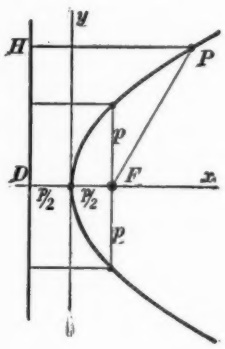
THE PARABOLA: $y^2 = 2px$.

21. DEFINITION: The locus of a point which moves so that

$$\frac{\text{its distance from a fixed point}}{\text{its distance from a fixed line}} = 1$$

is called a *parabola*.

The fixed point is called the *focus* and the fixed line the *directrix*. The line perpendicular to the directrix through the focus is called the *principal axis*. There is evidently only one point of the principal axis which is also a point of the curve, namely the point half way between the focus and the directrix; this point is called the *vertex*.



22. If we take the vertex as the origin and the principal axis as the axis of x, the *equation of the parabola* is

$$y^2 = 2px,$$

where $p =$ the distance between focus and directrix.*

23. The form of the curve is therefore that shown in the figure.† By definition $PF = PH$ for every point P on the curve.

The breadth of the curve at the focus is called the *latus rectum*, and is equal to $2p$.

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* Proof: If (x, y) is any point on the curve, then

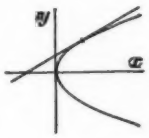
$$\sqrt{(x - \tfrac{1}{2}p)^2 + (y - 0)^2} = x + \tfrac{1}{2}p.$$

Many British authors write the equation in the form $y^2 = 4ax$, to avoid fractions. Other writers use $y^2 = 4px$ for the same purpose; this latter form, however, is unfortunate, since $2p$ is a fairly well-established notation for the latus rectum in each of the conics.

† Thus when x is 0, y is 0. When x increases, y increases, plus and minus; the curve is symmetrical with respect to the x-axis. When x is negative, y is imaginary. When $x = p/2$, $y = \pm p$; when $x = 2p$, $y = \pm 2p$.

24. To find the *equation of a tangent* to the parabola $y^2 = 2px$, use one of the following formulas:
when the point of contact, (x_1, y_1) is given:*

$$y_1 y = p(x + x_1);$$

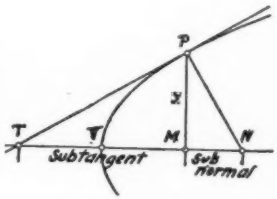


when the slope, m, is given:†

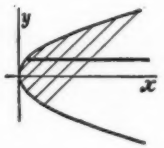
$$y = mx + \frac{p}{2m}.$$

A line perpendicular to a tangent at the point of contact is called a *normal*.

If the tangent and normal at any point P meet the principal axis at T and N, the projections of PT and PN on the principal axis are called the *subtangent* and *subnormal*, respectively. The subtangent is bisected by the vertex. The subnormal is constant, equal to the semi-latus rectum, p.



25. The locus of the middle points of a set of parallel chords in the parabola is a straight line parallel to the principal axis; such a line is called a *diameter*. In the parabola $y^2 = 2px$, if the slope of the parallel chords is m, then the equation of the diameter is $y = p/m$.‡



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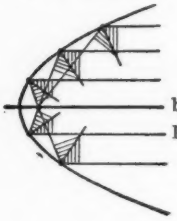
* Proof: Let $P_2 = (x_1 + h, y_1 + k)$ be a second point on the curve, near P_1; then the *slope of the tangent* at P_1 will be the limit of k/h as P_2 approaches P_1 along the curve, namely $m = p/y_1$. Then use § 11. The slope of the curve may also be found by the general method of the differential calculus.

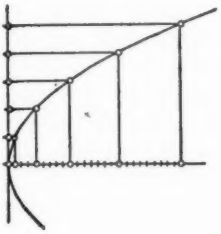
† Proof: Determine β so that the line $y = mx + \beta$ shall have only one point in common with the curve. [Remember that a quadratic equation $Ax^2 + Bx + C = 0$ will have equal roots if $B^2 - 4AC = 0$.]

‡ Proof: Let (x_0, y_0) be any point of the required locus; find the points of intersection of the curve and a line through (x_0, y_0) with slope m; then express the condition that (x_0, y_0) shall be the middle point between these two points. [Remember that the sum of the roots of a quadratic equation $Ax^2 + Bx + C = 0$ is $-B/A$.]

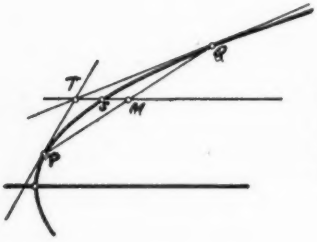
25a. Among the many properties of the parabola which should be worked out as problems, the following may be mentioned as especially important, and easy to remember:



1. The normal at any point P bisects the angle formed by the line from P to the focus and the line through P parallel to the principal axis (parabolic mirror).



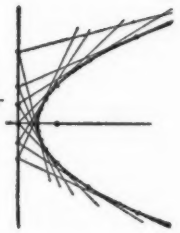
2. If P_1, P_2, ... are any points on a parabola, the distances of these points from the principal axis are proportional to the squares of their distances from the tangent at the vertex.



3. If the tangents at P and Q intersect at T, and if M is the middle point of the chord PQ, then the line through T and M is a diameter, and the segment TM is bisected by its point of intersection with the curve.

4. The locus of the foot of the perpendicular from the focus on a moving tangent is the tangent at the vertex.

5. The locus of the point of intersection of perpendicular tangents is the directrix.



Note. The usual methods for constructing a parabola should also be given.

CHAPTER V.

THE ELLIPSE: $b^2x^2 + a^2y^2 = a^2b^2$.

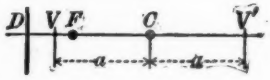
26. DEFINITION: The locus of a point which moves so that

$$\frac{\text{its distance from a fixed point}}{\text{its distance from a fixed line}} = e < 1$$

(where e is a constant less than 1), is called an *ellipse.*

The fixed point is called the *focus,* the fixed line the *directrix,* and the constant, e, the *eccentricity.* The line perpendicular to the directrix through the focus is called the *principal axis.* There are evidently two points of the principal axis which are also points of the curve; these two points are called the *vertices,* and the point half way between them is called the *centre.*

27. If we let $2a =$ the distance between the vertices, then:*



the distance between the *centre* and either *vertex* is a;
the distance between the *centre* and the *focus* is ae;
the distance between the *centre* and the *directrix* is a/e.

28. If we take the centre as the origin and the principal axis as the axis of x, the *equation of the ellipse* is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where b is an abbreviation for $a\sqrt{1 - e^2}$.† Note that $b < a$.

---

* Proof: Since the vertices, V and V', are points of the curve, $VF/VD = e$ and $V'F/V'D = e$; that is,

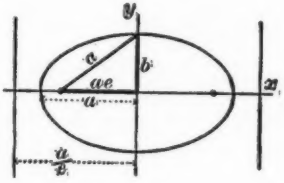
$$\frac{a - CF}{CD - a} = e \quad \text{and} \quad \frac{a + CF}{a + CD} = e,$$

whence $CF = ae$ and $CD = a/e$.

† Proof: If (x, y) is any point on the curve, then

$$\frac{\sqrt{(x + ae)^2 + (y - 0)^2}}{x + \frac{a}{e}} = e.$$

29. The form of the curve is therefore that shown in the figure.* The right triangle enables us to find any one of the three quantities a, b, and e, when the other two are given.



The symmetry of the equation shows that the curve might equally well have been obtained, with the same eccentricity, e, from a *second focus and directrix*, shown on the right.

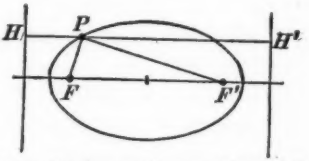
The breadth of the curve at either focus is called the *latus rectum*, and is equal to $2a(1 - e^2)$, or $2b^2/a$.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

30. Let P be any point of the ellipse, F and F' the foci, and PH and PH' the perpendiculars from P to the directrices; then

(1) $PF/PH = e$ and $PF'/PH' = e$,

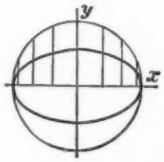
by definition of the curve. Furthermore:†



(2) $$PF + PF' = 2a.$$

In fact, the ellipse is often defined as *the locus of a point which moves so that the sum of its distances from two fixed points is constant.*

31. If a circle be described upon the major axis of an ellipse as diameter, each ordinate in the ellipse is to the corresponding ordinate in the circle as b is to a.‡ In fact, the ellipse is often defined as *the locus of points dividing the ordinates of a circle in a constant ratio.*



From this property it follows that the *area of an ellipse* is πab.

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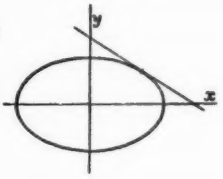
* Thus, when $y = 0$, $x = \pm a$; when $x = 0$, $y = \pm b$. The curve is symmetrical with respect to both axes. In first quadrant, as x increases, y decreases (slowly when x is small, and rapidly when x approaches a).

† Proof: $PF = e(PH)$ and $PF' = e(PH')$, so that $PF + PF' = e(HH') = e(2a/e) = 2a$.

‡Proof: In the ellipse, $y = \frac{b}{a}\sqrt{a^2 - x^2}$; in the circle, $y = \sqrt{a^2 - x^2}$.

32. To find the *equation of a tangent* to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, use one of the following formulas:*



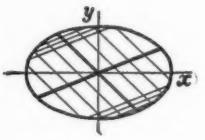
when the point of contact, (x_1, y_1), is given:

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1;$$

when the slope, m, is given:

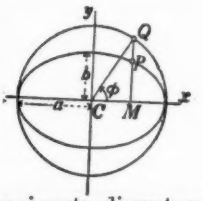
$$y = mx \pm \sqrt{a^2 m^2 + b^2}.$$

33. The locus of the middle points of a set of parallel chords in the ellipse is a straight line through the centre; such a line is called a *diameter*. In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if the slope of the parallel chords is m, then the slope of the diameter is $-\frac{b^2}{a^2 m}$.*



Any two lines through the centre, such that the product of their slopes is $-b^2/a^2$, are called a pair of *conjugate diameters*, because each bisects all chords parallel to the other.

34. The circle described in § 31 is called the *auxiliary circle*. If P is any point on the ellipse, and Q the corresponding point on the auxiliary circle (see figure), then the angle ϕ which CQ makes with the axis is called the *eccentric angle* of the point P. From the figure, and § 31,



$$x = a\cos\phi \quad \text{and} \quad y = b\sin\phi,$$

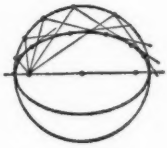
where x, y are the coordinates of P.

The eccentric angles of the ends of two conjugate diameters differ by 90°.

* Proof as in the case of the parabola.

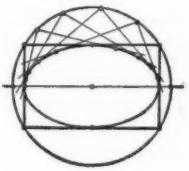
34a. Among the many properties of the ellipse which should be worked out as problems, the following are especially easy to remember:

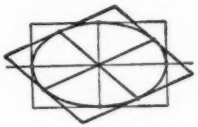
1. The normal at any point P bisects the angle formed by the lines joining P with the foci.



2. The locus of the foot of the perpendicular from the focus on a moving tangent is the circle on the major axis as diameter.

3. The locus of the point of intersection of perpendicular tangents is a circle with radius $\sqrt{a^2 + b^2}$.





4. The area of a parallelogram bounded by tangents parallel to conjugate diameters is constant.

Note. The usual methods for constructing an ellipse should also be given.

CHAPTER VI.

THE HYPERBOLA: $b^2x^2 - a^2y^2 = a^2b^2$.

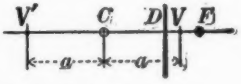
35. DEFINITION: The locus of a point which moves so that

$$\frac{\text{its distance from a fixed point}}{\text{its distance from a fixed line}} = e > 1$$

(where e is a constant greater than 1), is called a *hyperbola.*

The fixed point is called the *focus*, the fixed line the *directrix*, and the constant, e, the *eccentricity.* The line perpendicular to the directrix through the focus is called the *principal axis.* There are evidently two points of the principal axis which are also points of the curve; these two points are called the *vertices*, and the point half way between them is called the *centre.*

36. If we let $2a =$ the distance between the vertices, then:*



the distance between the *centre* and either *vertex* is a;
the distance between the *centre* and the *focus* is ae;
the distance between the *centre* and the *directrix* is a/e.

37. If we take the centre as the origin and the principal axis as the axis of x, the *equation of the hyperbola* is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where b is an abbreviation for $a\sqrt{e^2 - 1}$.† Note that b may be greater or less than a, or equal to a.

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* Proof: Since the vertices, V and V', are points of the curve, $VF/VD = e$ and $V'F/V'D = e$; that is,

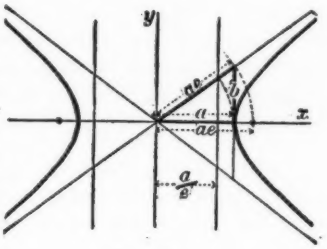
$$\frac{CF - a}{a - CD} = e \quad \text{and} \quad \frac{a + CF}{a + CD} = e,$$

whence $CF = ae$ and $CD = a/e$.

† Proof: If (x, y) is any point on the curve, then

$$\frac{\sqrt{(x - ae)^2 + (y - 0)^2}}{x - \frac{a}{e}} = e.$$

38. The form of the curve is therefore that shown in the figure.* The two lines through the centre with slopes $\pm b/a$ are called the *asymptotes* of the hyperbola; the two branches of the curve approach these lines more and more nearly as they recede from the centre.† The right triangle enables us to find any one of the three quantities, a, b, and e, when the other two are given.

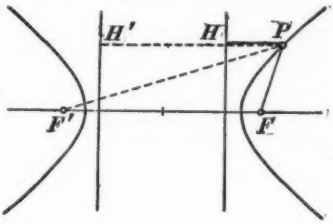


$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The symmetry of the equation shows that the curve might equally well have been obtained, with the same eccentricity, e, from a *second focus and directrix*, shown on the left.

The breadth of the curve at either focus is called the *latus rectum*, and is equal to $2a(e^2 - 1)$, or $2b^2/a$.

39. Let P be any point of the hyperbola, F and F' the foci, and PH and PH' the perpendiculars from P to the directrices; then



(1) $PF/PH = e$ and $PF'/PH' = e$,

by the definition of the curve. Furthermore:‡

(2) $|\ PF - PF'\ | = 2a.$

In fact, the hyperbola is often defined as the *locus of a point which moves so that the difference of its distances from two fixed points is constant.*

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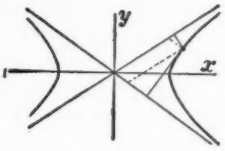
* Thus, when $y = 0, x = \pm a$; when $x = 0$, or $x < a$, y is imaginary; when x increases beyond a, y increases, plus and minus (most rapidly when x is little greater than a). The curve is symmetrical with respect to both axes.

† For, the slope $\frac{y}{x} = \frac{b}{a}\sqrt{1 - \frac{a^2}{x^2}}$ approaches $\frac{b}{a}$ as x increases; moreover, if y_1 is the ordinate of any point on the curve, and y_2 the ordinate of the corresponding point on the asymptote, then the difference $y_2 - y_1$ approaches zero; for $y_2^2 - y_1^2 = b^2$, and therefore $y_2 - y_1 = b^2/(y_2 + y_1)$.

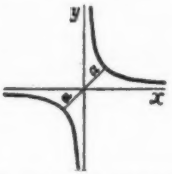
‡ Proof: $PF = e(PH)$ and $PF = e(PH')$, so that $|\ PF - PF'\ | = e(HH') = e\,(2a/e) = 2a$.

40. The product of the distances from any point of a hyperbola to the asymptotes is constant. Hence, the hyperbola is often defined as *the locus of a point which moves so that the product of its distances from two fixed lines is constant.* (The distances here may be the perpendicular distances; or, the distance to each line may be measured parallel to the other.)



41. An important special case is that of the "rectangular" hyperbola, whose asymptotes are perpendicular ($a = b$); the *equation of the rectangular hyperbola referred to its asymptotes as axes* is (by § 40)

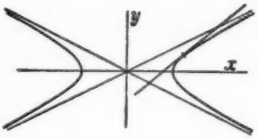
$$xy = \frac{a^2}{2}.$$



42. To find the *equation of a tangent* to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, use one of the following formulas:*

when the point of contact, $(x_1 y_1)$, is given:

$$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1;$$



when the slope, m, is given

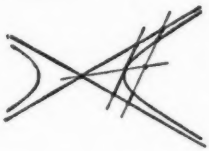
$$y = mx \pm \sqrt{a^2 m^2 - b^2}.$$

43. The locus of the middle points of a set of parallel chords in the hyperbola is a straight line through the centre; such a line is called a *diameter*. In the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if the slope of the parallel chords is m, the slope of the diameter will be $\frac{b^2}{a^2 m}$.*

Any two lines through the centre, such that the product of their slopes is b^2/a^2, are called a pair of *conjugate diameters*, because each bisects all chords parallel to the other.

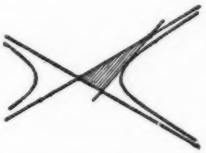
* Proof as in the case of the parabola.

43a. Among the properties of the hyperbola, the following are easy to remember:



1. If a line cuts the hyperbola and its asymptotes, the parts of the line intercepted between the curve and the asymptotes are equal. In particular, the portion of any tangent intercepted between the asymptotes is bisected by the point of tangency.

2. The area of the triangle bounded by any tangent and the asymptotes is constant.



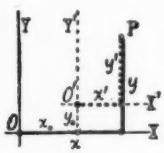
Note. The usual methods of constructing a hyperbola—especially the rectangular hyperbola—should be given.

CHAPTER VII.

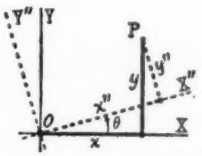
TRANSFORMATION OF COÖRDINATES.*

44. The equation of a curve can often be simplified by a "changes of axes," either changing to a new origin (x_0, y_0), or turning the axes through an angle θ, or both.

If (x, y), (x', y'), (x'', y'') are the coördinates of the same point P, referred to three different sets of axes, as in the figure, then



$$x = x_0 + x'$$
$$y = y_0 + y'$$



$$x = x'' \cos\theta - y'' \sin\theta \dagger$$
$$y = x'' \sin\theta + y'' \cos\theta$$

Suppose now that the point P is allowed to move under certain conditions given by an equation in x and y. The same condition can be expressed in terms of x' and y' or in terms of x'' and y'' by substituting in the given equation the values of x and y just found. This process is called a transformation of coördinates, from the axes x, y to the axes x', y', or to the axes x'', y''; and the new equation can often be made simpler than the given equation by a suitable choice of x_0 and y_0, or θ.

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* See also the chapter on polar coördinates.

† These last formulas are most easily remembered as follows:

x = easterly displacement of P,
= (easterly component of x'') + (easterly component of y'')
$= x'' \cos\theta - y'' \sin\theta$,

y = northerly displacement of P
= (northerly component of x'') + (northerly component of y'')
$= x'' \sin\theta + y'' \cos\theta$.

CHAPTER VIII.

GENERAL EQUATION OF THE SECOND DEGREE IN x AND y.

45. The general equation of the second degree in x and y is of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

By a suitable transformation of coördinates this equation can always be brought into one or other of the following forms:

$$A'x^2 + C'y^2 + F' = 0, \quad C''y^2 + D''x = 0, \quad C''y^2 + F''' = 0,$$

and hence can be shown to represent a conic section, using this term in a general sense to include (1) an ellipse, which may be real, null, or imaginary; (2) a hyperbola, or a pair of intersecting lines; (3) a parabola, or a pair of parallel lines (distinct, coincident, or imaginary).*

The student should be able to plot readily the locus of an equation of the second degree in any of the simple cases mentioned below—these being the cases which occur most often in practice.

46. To plot $Ax^2 + Cy^2 + F = 0$, where A and C have the same sign. Find the intercepts on the axes (by putting $x = 0$ and $y = 0$); if both are real, we have an ellipse in which $a =$ the larger of the two intercepts, and $b =$ the smaller; or if $A = C$, the ellipse becomes a circle. If both intercepts are zero, or imaginary, the locus is a single point, or imaginary.

* Proof: If $B^2 - 4AC$ is not zero, transform to parallel axes with origin at (x_0, y_0), and choose x_0 and y_0 so that the terms of the first degree in the new equation shall vanish; then turn the axes through an angle θ, and choose θ so that the term in xy shall vanish. If $B^2 - 4AC = 0$, turn the axes through an angle θ, and choose θ so that the term in xy shall vanish; then transform to a new origin (x_0, y_0), and choose x_0 and y_0 so that the constant term and one of the terms of the first degree, or so that both the terms of the first degree shall vanish. For special methods of abbreviating the computation in numerical cases see § 55, note.

47. To plot $Ax^2 + Cy^2 + F = 0$, where A and C have opposite signs. Unless $F = 0$, one of the intercepts will be real and the other imaginary, and the curve will be a hyperbola whose principal axis is the axis on which the intercepts are real. To find the slopes of the asymptotes, divide by x^2 and find the limit of y/x as x increases indefinitely. If $F = 0$, the locus is a pair of intersecting lines.

48. To plot $Cy^2 + Dx + F = 0$. Write this as

$$y^2 = -\frac{D}{C}\left(x + \frac{F}{D}\right), \quad \text{or} \quad y^2 = -\frac{D}{C}x'.$$

This is a parabola with vertex at $x_0 = -F/D$, and running out along the positive or negative axis of x. Plotting one or two points will fix the direction, and comparison with the equation $y^2 = 2px$ will give the semi-latus rectum, p.

49. To plot $Ax^2 + Cy^2 + Dx + Ey + F = 0$. Write this in the form

$$A\left(x^2 + \frac{D}{A}x + \quad\right) + C\left(y^2 + \frac{E}{C}y + \quad\right) = -F,$$

and "complete the squares"; then reduce to the form $Ax'^2 + Cy'^2 + F = 0$ by an obvious change of origin.

50. To plot $Cy^2 + Dx + Ey + F = 0$. Complete the square of the terms in y and reduce to the form $Cy^2 + Dx + F = 0$ by an obvious change of origin.

51. To plot $Bxy + F = 0$. This is a rectangular hyberbola referred to its asymptotes (see § 41). The equation $Bxy + Dx + Ey + F = 0$ can be reduced to this form by moving the origin to $x_0 = -E/B$, $y_0 = -D/B$.

52. If the equation to be plotted does not come under any of the forms just considered, a fair idea of the position of the curve may be found by the following very elementary method. Solving the equation for y in terms of x, we have, if C is not zero,

$$y = -\frac{Bx + E}{2C} \pm \frac{1}{2C}\sqrt{X},$$

where X is an expression involving x alone. Finding the

values of $-(Bx + E)/2C$, and adding and subtracting the values of $\sqrt{X}/2C$, for various values of x, we can find as many points (x, y) on the curve as we please. Or, again, solving for x in terms of y, we have, if A is not zero,

$$x = -\frac{By + D}{2A} \pm \frac{1}{2A}\sqrt{Y},$$

where Y is an expression involving y alone. From this equation we can find values of x corresponding to as many values of y as we please.

This method is very easy to remember, but does not give readily the exact dimensions of the curve.

53. The center of the curve will be the point of intersection of the two lines

$$2Ax + By + D = 0,$$
$$Bx + 2Cy + E = 0,$$

except when $B^2 - 4AC = 0$, in which case these lines will be parallel, and the curve has no center.

54. The slopes of the lines joining the origin with the infinitely distant points of the curve (if any) are given by writing the terms of the second degree equal to zero:

$$Ax^2 + Bxy + Cy^2 = 0,$$

dividing through by x^2 (or y^2), and solving for y/x (or x/y).

55. If a more detailed discussion of the curve is required, it is best to follow the special methods of reduction given in the text-books (compare foot-note in § 45).†

56. The student should be familiar with the geometric proof that all the "conic sections" can be obtained as plane

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* The student of the calculus will recognize these equations as

$$\partial F/\partial x = 0 \quad \text{and} \quad \partial F/\partial y = 0,$$

where

$$F(x, y) \equiv Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is the equation of the curve.

† The resulting formulæ are given here for reference, although the problem is not one of common occurrence.

Required, to plot the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

sections of a right circular cone. It is a profitable exercise to construct a cone, given the vertex and a hyperbolic section. It should also be made thoroughly clear why an elliptic section is a symmetrical figure instead of egg-shaped.

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* Case I. *Central conic.* If $B^2 - 4AC$ is not zero, transform to the center as a new origin:

$x_0 = (2CD - BE)/(B^2 - 4AC)$, $y_0 = (2AE - BD)/(B^2 - 4AC)$;

then turn the axes through a positive acute angle θ given by

$$\tan 2\theta = B/(A - C).$$

The transformed equation will be

$$A'x^2 + C'y^2 + F' = 0,$$

where $F' = Dx_0/2 + Ey_0/2 + F$, while A' and C' are found by solving the equations $A' + C' = A + C$, $A' - C' = \pm\sqrt{(A - C)^2 + B^2}$, where the sign before the radical is to be + or − acording as B is positive or negative. The reduced equation can be plotted as in §§ 46, 47.

Case II. *Parabolic type.* If $B^2 - 4AC = 0$, the equation may be written in the form $(ax + cy)^2 + Dx + Ey + F = 0$, where $a = \sqrt{A}$ while $c = \sqrt{C}$ or $c = -\sqrt{C}$ according as B is positive or negative. The locus will be of the parabolic type. Take as a new axis of x' the line

$$ax + cy + m = 0,$$

where $m = (aD + cE)/2(A + C)$, and choose the positive direction along this line so that it shall make a (positive or negative) acute angle with the axis of x. This line will be the principal axis of the curve.

Two subcases may now occur.

(a) If a/c is not equal to D/E, take as axis of y' the line

$$cx - ay + n = 0,$$

where $n = (A + C)\ (m^2 - F)/(aE - cD)$. This line will be the tangent at the vertex, and the transformed equation will be

$$y' = 2px',$$

where $2p = (cD - aE)/\sqrt{(A + C)^3}$. The locus is a true parabola.

(b) If $a/c = D/E$, the equation referred to the axis of x' will be

$$y' = m^2 - F,$$

which represents a pair of distinct, coincident, or imaginary parallel lines.

CHAPTER IX.

SYSTEMS OF CONICS.

57. If U and V are expressions of the second degree in x and y, the equations $U=0$ and $V=0$ will represent conics; then (a) the equation $U+kV=0$, where k is any constant, will represent another conic passing through all the points of intersection of the first two, and having no other points in common with either of them; and (b) the equation $UV=0$ will represent a curve made up of the two conics $U=0$ and $V=0$ taken together. The same theorems hold good if U and V are any expressions in x and y (not necessarily of the second degree).

58. To find the equation of a conic through five points, let $u=0$ and $v=0$ be the equations of the lines P_1P_2 and P_3P_4, and let $u'=0$ and $v'=0$ be the equations of the lines P_1P_3 and P_2P_4. Then $uv+ku'v'=0$, where k is any constant, will be the equation of a conic through these four points. It remains to determine k so that this conic shall pass through P_5.

59. The equation

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}=1,$$

where k is an arbitrary constant, represents a family of confocal ellipses and hyperbolas, which intersect at right angles.

CHAPTER X.

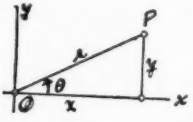
POLAR COÖRDINATES.

This chapter, placed here for convenience of reference, may well be introduced, in teaching, much earlier in the course.

60. It is often convenient to represent the position of a point P by giving the angle, ϕ, which the line through O and P makes with the x-axis, and the distance, r, from O to P along this line. The angle ϕ is called the *vectorial angle*, or simply the *angle*, of the point P, and is measured from the positive direction of the axis of x to the positive direction of the line through O and P. The distance $r = OP$ is called the *radius vector* of the point P, and is positive or negative according as it runs forward or backward along the line through O and P.

It is customary to take r positive, and let ϕ range from 0° to 360°.

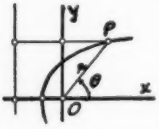
61. From the figure,



$$x = r\cos\phi, \quad y = r\sin\phi,$$
$$x^2 + y^2 = r^2, \quad y/x = \tan\phi.$$

By the aid of these relations, we can transform any equation from rectangular to polar coördinates, and vice versa.

62. The polar equation of a conic, referred to the focus as origin, and the principal axis as axis of x (see figure) is



$$r = \frac{p}{1 - e\cos\theta}$$

where p is the semi-latus rectum, and e the eccentricity.

63. Plotting curves in polar coördinates is an excellent exercise in reviewing the trigonometric functions. The work should be so arranged that no critical value of the function occurs between two successive assigned values of θ.

CHAPTER XI.

Coordinates in Space.

64. Four methods are in use for representing numerically the position of a point in space. If Ox, Oy, Oz are three mutually perpendicular axes, the position of any point P may be determined by:

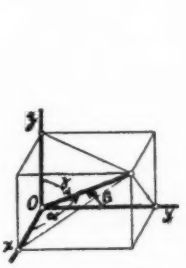
(1) Rectangular coördinates, x, y, z;

(2) Polar coördinates in space, r, α, β, γ, where the angles α, β, γ are subject to the restriction $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$;

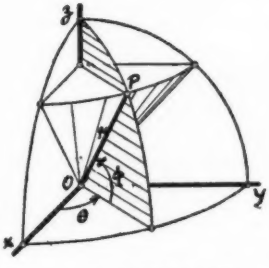
(3) Spherical coördinates, r, ϕ, θ, where $\phi =$ the latitude of P, and θ its longitude:

(4) Cylindrical coördinates, ρ, θ, z.

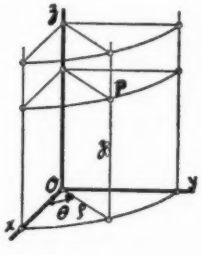
The relations between the various sets of coördinates are as follows:



$x = r\cos\alpha,$
$y = r\cos\beta,$
$z = r\cos\gamma.$



$x = r\cos\phi\cos\theta$
$y = r\cos\phi\sin\theta$
$z = r\sin\theta$



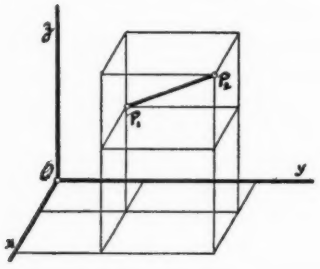
$\rho = r\cos\phi,$
$\rho^2 = x^2 + y^2,$
$x^2 + y^2 + z^2 = r^2.$

As there is no well-established uniformity in the use of the letters in spherical coördinates, or in the choice of the positive directions along the axes, it is important, in reading any

author, to note, on a figure, the exact meanings of the letters he employs.

65. Distance between two points, in terms of their coördinates:

$$P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$



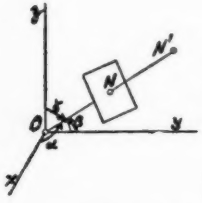
66. Angle ψ between two lines whose direction cosines are given:

$$\cos\psi = l_1l_2 + m_1m_2 + n_1n_2,$$

where $l_1 = \cos\alpha_1$, $m_1 = \cos\beta_1$, $n_1 = \cos\gamma_1$, etc.*

67. Equation of a plane:

$$lx + my + nz = p,$$



where p = perpendicular distance from the origin, and l, m, n = the direction cosines of the normal to the plane.*

Every equation of the form $Ax + By + Cz + D = 0$ represents a plane; for, it can be thrown into the form $lx + my + nz = p$ by dividing through by $\sqrt{A^2 + B^2 + C^2}$.

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* Proof: let (1) and (2) be lines through the origin, parallel to the given lines; on these lines take points P_1, P_2 at a distance r from the origin; then

$P_1P_2^2 = r^2 + r^2 - 2rr\cos\psi = (rl_1 - rl_2)^2 + (rm_1 - rm_2)^2 + (rn_1 - rn_2)^2$.

* Proof: The foot of the perpendicular is $N = (pl, pm, pn)$; take $N' = (2pl, 2pm, 2pn)$ and express the condition that the point (x, y, z) shall be equidistant from O and N'.

68. Equation of sphere with center at the origin:

$$x^2 + y^2 + z^2 = r^2.$$

69. Equation of ellipsoid, with center at the origin:

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

70. Any equation in x, y, z will represent a surface (real or imaginary), the form of which can be investigated by the *method of plane sections.* Thus, putting $x = x_1$, the equation becomes an equation in y and z, which represents a curve in the plane $x = x_1$; similarly for $y = y_1$ and $z = z_1$.

71. Any equation of the second degree in x, y, z represents a (real or imaginary) surface of the second degree, or *conicoid.* The types of real conicoids are as follows:

(1) *Ellipsoid,* with semi-axes a, b, c. Special case: ellipsoid of revolution, generated by rotating an ellipse about its major axis (prolate spheroid) or about its minor axis (oblate spheroid).

(2) *Hyperboloid of two sheets.* .Special case: generated by rotating a hyperbola about its principal axis.

(3) *Hyperboloid of one sheet,* or *ruled hyperboloid.* Special case: generated by rotating a hyperbola about its conjugate axis. Two sets of straight lines can be drawn on this surface.

(4) *Elliptic paraboloid.* Special case: generated by rotating a parabola about its principal axis.

(5) *Hyperbolic paraboloid,* or *ruled paraboloid.* A saddle-shaped figure, on which two sets of straight lines can be drawn.

(6) *Cone,* generated by a straight line always passing through a fixed point called the vertex, and always touching a fixed conic, called the directrix. If the directrix is a circle, the cone is a circular cone (right or oblique). If the vertex recedes to infinity, the cone becomes a cylinder. On any cone a single set of straight lines can be drawn.

The student should become familiar with at least the shapes of these surfaces, through diagrams or models.

Any plane section of any surface of the second degree is a conic.